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A systematic study on the exact solution of the position dependent mass Schrödinger equation

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Abstract
An algebraic method of constructing potentials for which the Schrödinger equation with position dependent mass can be solved exactly is presented. A general form of the generators of *su*(1,1) algebra has been employed with a unified approach to the problem. Our systematic approach reproduces a number of earlier results and also leads to some novelties. We show that the solutions of the Schrödinger equation with position dependent mass are free from the choice of parameters for position dependent mass. Two classes of potentials are constructed that include almost all exactly solvable potentials.

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1. Introduction

The study of position dependent mass (PDM) Schrödinger equation has recently attracted some interest [1, 2] arising from the study of electronic properties of semiconductors, liquid crystals, quantum dots, and the recent progress of crystal-growth techniques for production of non-uniform semiconductor specimen in which carrier effective mass depends on position [3]. It is obvious that the study of the PDM Schrödinger equation has considerable impact on condensed matter physics as well as related fields of physics.

Exact solvability of the Schrödinger equation with constant mass has been the main interest since the early days of quantum mechanics [4]. It has been solved exactly for a large number of potentials by employing various techniques. In fact, for exactly solvable potentials, its general solution can be obtained in terms of some special functions by transforming the original Schrödinger equation into the second order differential equation . Systematic studies of these transformations have been given in [5] regarding the confluent hypergeometric functions. The relations between the algebraic technique and the special function theory have been discussed in [6]. Recently various approaches have been presented in a unified way and a number of earlier results have been reproduced [4]. In the present work we use the Lie algebraic technique to construct the Hamiltonian for the PDM Schrödinger equation and obtain the solutions in terms of the special functions.

In the PDM Schrödinger equation, the mass and momentum operator no longer commute. The general expression for the kinetic energy operator was introduced by von Roos [7]:

$$T = \tfrac{1}{4}(m^{\eta}\mathbf{p}m^{\varepsilon}\mathbf{p}m^{\rho} + m^{\rho}\mathbf{p}m^{\varepsilon}\mathbf{p}m^{\eta}) \tag{1}$$

where $\eta + \varepsilon + \rho = -1$ is a constraint. One of the problems is the choice of parameters [8, 9]. In our approach we obtain an exact solution of the PDM Schrödinger equation without any particular choice which leads to a general solution where the choice of the parameters distinguishes the physical systems.

One can display a number of fruitful applications of the Lie algebraic technique, in particular, in atomic and nuclear physics and other fields of physics. Our task here is to obtain the exact solution of the PDM Schrödinger equation by the use of the $su(1,1)$ algebra technique.

The paper is organized as follows. In section 2 we present a general Hamiltonian by using $su(1,1)$ algebra and discuss its relation with the PDM Schrödinger equation. We obtain a general expression for the potential. In section 3 we describe the application of the $su(1,1)$ algebra to obtain Coulomb, harmonic oscillator and Morse family potentials. In section 4 we construct hyperbolic and trigonometric potentials. Finally we discuss our results in section 5.

2. Structure of the *su*(1,1) Lie algebra and PDM Schrödinger equation

The Lie algebraic technique is suitable for studying the PDM Schrödinger equation, because it contains a first-derivative term. The $su(1,1)$ Lie algebra is described by the commutation relations,

$$[J_+, J_-] = -2J_0 \qquad [J_0, J_\pm] = \pm J_\pm. \tag{2}$$

The Casimir operator of this structure is given by

$$J^2 = -J_\pm J_\mp + J_0^2 \mp J_0. \tag{3}$$

The eigenstate of J^2 and J_0 can be denoted by $|jN\rangle$ where

$$J^2|jN\rangle = j(j+1)|jN\rangle \qquad J_0|jN\rangle = -N|jN\rangle \tag{4}$$

while the allowed values of N are

$$N = -j, -j+1, -j+2, \ldots = (-j+n) \tag{5}$$

where n is a positive integer. We consider the most general form of the generators of the algebra which was introduced by Sukumar [10]

$$\begin{aligned} J_\pm &= \mathrm{e}^{\pm\mathrm{i}\phi}\left(\pm h(x)\frac{\partial}{\partial x} \pm g(x) + f(x)J_0 + c(x)\right) \\ J_0 &= -\mathrm{i}\frac{\partial}{\partial\phi}. \end{aligned} \tag{6}$$

The commutation relations (2) are satisfied when the functions $h(x)$, $f(x)$ and $c(x)$ take the forms

$$h(x) = \frac{r}{r'} \qquad f(x) = \frac{1+ar^2}{1-ar^2} \qquad c(x) = -\frac{br}{1-ar^2} \tag{7}$$

where $r = r(x)$ and a and b are constants. The differential realization (6) can be used to derive the second order differential equations of the orthogonal polynomials. The differential equations of these polynomials can be expressed in terms of Casimir operator J^2:

$$H = J^2 \qquad H|jN\rangle = j(j+1)|jN\rangle. \tag{8}$$

Let us consider the basis function,

$$|jN\rangle = \mathrm{e}^{-\mathrm{i}N\phi}\Re_{jN}(x). \tag{9}$$

In terms of the realizations (6) and with the basis (9), the Hamiltonian (8) takes the form

$$\begin{aligned} H = {} & \frac{r^2}{r'^2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{r}{r'}\left(2g - \frac{2ar^2}{1-ar^2} - \frac{rr''}{r'^2}\right)\frac{\mathrm{d}}{\mathrm{d}x} \\ & + g^2 + g + \frac{rg'}{r'} - \frac{2g}{1-ar^2} - \frac{r(2N+br)(2aNr+b)}{(1-ar^2)^2}. \end{aligned} \tag{10}$$

Let us now turn our attention to the PDM Schrödinger equation which can be written as

$$H_{\mathrm{PDM}} = T + V(x) \qquad H_{\mathrm{PDM}}\psi(x) = E\psi(x) \tag{11}$$

where $V(x)$ is the potential of the physical system and $\psi(x)$ and E are eigenstates and eigenvalues of the PDM Schrödinger equation. Introducing the eigenfunction and momentum operator p

$$\psi(x) = -\frac{2mr^2}{r'^2}\Re(x) \qquad p = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{dx}} \tag{12}$$

respectively, the position dependent mass Hamiltonian takes the form

$$\begin{aligned} H_{\mathrm{PDM}} = {} & \frac{r^2}{r'^2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{r}{r'}\left(4 - \frac{4rr''}{r'^2} + \frac{rm'}{r'm}\right)\frac{\mathrm{d}}{\mathrm{d}x} + 2 + \frac{2r}{r'^2}\left(\frac{3rr''^2}{r'^2} - \frac{rr'''}{r'} - 3r''\right) \\ & + \frac{m'r^2}{mr'^2}\left(\frac{(1+\eta)(\varepsilon+\eta)m'}{m} + \frac{(1-\varepsilon)m''}{2m'} + \frac{2(r'^2 - rr'')}{rr'}\right) - \frac{2mr^2}{r'^2}V(x) \end{aligned} \tag{13}$$

then comparing (13), (8) and (10), we obtain the following general expression for the potential,

$$\begin{aligned} V(x) - E = {} & \frac{(2bN + r(b^2 + a(4N^2-1) + 2abNr))r'^2}{2mr(1-ar^2)^2} \\ & + \frac{(j(j+1))r'^2}{2mr^2} + \frac{3r''^2}{8mr'^2} - \frac{r'''}{4mr'} + V_m(x) \end{aligned} \tag{14}$$

where $V_m(x)$ is given by

$$V_m(x) = \frac{1}{4m^2}\left(\frac{(4\varepsilon(1+\eta) + (1+2\eta)^2)m'^2}{2m} - \varepsilon m''\right) \tag{15}$$

when the function $g(x)$ constrained to

$$g(x) = \frac{ar^2-2}{ar^2-1} + \frac{m'r}{2mr'} - \frac{3rr''}{2r'^2}. \tag{16}$$

We have so far constructed a class of position dependent mass potentials which reduces to the Natanzon class potentials for the constant mass. In this construction it has been emphasized [11] that for the exactly solvable cases, the energy levels form an infinite sequence by fixing

j and varying N, such that n takes values $0, 1, \ldots, \infty$ and one can obtain the full spectrum of the Hamiltonian (13). But in the quasi-exactly solvable potentials [11, 12], there exist at most $N + 1$ levels for each choice of N which can exactly be obtained. In the following section we construct the quantum mechanical potentials.

3. Coulomb, harmonic oscillator and Morse family potentials

In order to obtain the corresponding potentials we choose $a = 0$, then the potential (14) takes the form

$$V(x) - E = \left(\frac{b^2}{2} + \frac{j(j+1)}{2r^2} + \frac{bN}{r}\right)\frac{r'^2}{m} + \frac{3r''^2}{8mr'^2} - \frac{r'''}{4mr'} + V_m(x). \qquad (17)$$

In the above potential, the energy term on the left-hand side should be represented by a constant term on the right-hand side. This condition can be satisfied when

$$(\lambda_0 + \lambda_1 r^{-1} + \lambda_2 r^{-2})\frac{r'^2}{m} = 1 \qquad (18)$$

where λ_0, λ_1 and λ_2 are constants. Choosing appropriate values of λ_0, λ_1 and λ_2, one can generate quantum mechanical potentials.

3.1. Coulomb family potentials

In order to generate Coulomb family potentials, we choose $\lambda_0 = 1$, and $\lambda_1 = \lambda_2 = 0$. Solving (18) for r and substituting into (17), we obtain the following potential

$$V(x) = \frac{j(j+1)}{2u^2} + \frac{Ze^2}{u} + U_m(x) \qquad (19)$$

with the eigenvalues

$$E = -\frac{Z^2e^4}{2N^2} \qquad (20)$$

where $u = \int_0^x \sqrt{m}\,\mathrm{d}x$, $N = -j + n$ and the parameter b of the potential (17) is defined as $b = Ze^2/N$. The potential is isospectral with the constant mass Schrödinger equation. The mass dependent function $U_m(x)$ is given by

$$U_m(x) = \frac{m'}{8m^2}\left(\frac{5m'}{4m} - \frac{m''}{m'}\right) + V_m(x). \qquad (21)$$

3.2. Harmonic oscillator potential

The harmonic oscillator potential can be generated from (17), under the condition $\lambda_1 = 1/2$, and $\lambda_0 = \lambda_2 = 0$. In this case $r = \frac{u^2}{2}$, and the potential takes the form

$$V = \frac{(1+4j)(3+4j)}{8u^2} + \frac{b^2}{2}u^2 + U_m(x) \qquad (22)$$

with the eigenvalues

$$E = -2bN \qquad (23)$$

3.3. Morse family potential

Our last example in this class of potentials is the Morse family potential. This potential can be obtained by setting parameters $\lambda_2 = 1/\alpha^2$, and $\lambda_0 = \lambda_1 = 0$. Solving (18) for r we obtain $r = \mathrm{e}^{-\alpha u}$ and the potential takes the form

$$V(x) = Nb\alpha^2\,\mathrm{e}^{-u} + \frac{b^2\alpha^2}{2}\,\mathrm{e}^{-2\alpha u} + U_m(x) \tag{24}$$

with the eigenvalues

$$E = -\frac{\alpha^2}{8}(1+2j)^2. \tag{25}$$

4. Hyperbolic and trigonometric potentials

In this section we construct hyperbolic and trigonometric potentials. Some of these potentials have important applications in condensed matter phenomena because of their periodicity. As we mentioned before, in the potential (14) a constant term should be represented with the energy term. We discuss below the problem for various potentials.

4.1. Pöschl–Teller family potential

For the choice of $r = \mathrm{e}^{-2\alpha u}$, $a = 1$, the result is

$$\begin{aligned} V(x) = {} & \frac{\alpha^2}{8}(b+2N-1)(b+2N+1)\,\mathrm{cosech}^2\alpha u \\ & - \frac{\alpha^2}{8}(b-2N-1)(b-2N+1)\,\mathrm{sech}^2\,\alpha u + U_m(x) \end{aligned} \tag{26a}$$

$$E = -\frac{\alpha^2}{2}(1+2j)^2 \tag{26b}$$

which is the Pöschl–Teller potential. The function u is given by

$$u = \int_0^x \sqrt{m}\,\mathrm{d}x. \tag{27}$$

For the given mass term, u should be integrable. The trigonometric form of the Pöschl–Teller potential can be obtained by substituting $\alpha \to \mathrm{i}\alpha$. In this case the potential and its eigenvalues are given by

$$\begin{aligned} V(x) = {} & \frac{\alpha^2}{8}(b+2N-1)(b+2N+1)\,\mathrm{cosec}^2\alpha u \\ & + \frac{\alpha^2}{8}(b-2N-1)(b-2N+1)\,\sec^2\alpha u + U_m(x) \end{aligned} \tag{28a}$$

$$E = \frac{\alpha^2}{2}(1+2j)^2. \tag{28b}$$

4.2. Generalized Pöschl–Teller family potential

In order to construct the generalized Pöschl–Teller family potential, we introduce

$$r = \mathrm{e}^{-\alpha u} \qquad a = 1. \tag{29}$$

Substituting r into (14), the resulting potential and corresponding eigenvalues read as

$$V(x) = \frac{\alpha^2}{8}(b^2 + 4N^2 - 1)\,\mathrm{cosech}^2\alpha u + \frac{\alpha^2}{2} bN \coth\alpha u\,\mathrm{cosech}\,\alpha u + U_m(x) \tag{30a}$$

$$E = -\frac{\alpha^2}{8}(1 + 2j)^2 \tag{30b}$$

The trigonometric form of this potential can be obtained replacing α by $\mathrm{i}\alpha$. Then the potential is given by

$$V(x) = \frac{\alpha^2}{8}(b^2 + 4N^2 - 1)\,\mathrm{cosec}^2\alpha u + \frac{\alpha^2}{2} bN \cot\alpha u\,\mathrm{cosec}\,\alpha u + U_m(x) \tag{31a}$$

$$E = \frac{\alpha^2}{8}((1 + 2j)^2). \tag{31b}$$

4.3. Scarf family potential

Let us now construct another potential by substituting $r = \mathrm{i}\,\mathrm{e}^{-\alpha u}$, $a = 1$ into equation (14). In this case we obtain the PT symmetric [13] Scarf family potential that has been obtained by Bagchi and Roychoudhury [14],

$$V(x) = -\frac{\alpha^2}{8}(b^2 + 4N^2 - 1)\,\mathrm{sech}^2\,\alpha u + \frac{\mathrm{i}\alpha^2}{2} bN\,\mathrm{sech}\,\alpha u \tanh\alpha u + U_m(x) \tag{32a}$$

$$E = -\frac{\alpha^2}{8}((1 + 2j)^2). \tag{32b}$$

When we replace $b \to \mathrm{i}b$, the potential becomes the Scarf family potential. When we replace α by $\mathrm{i}\alpha$, the potential becomes the trigonometric Scarf family potential,

$$V(x) = \frac{\alpha^2}{8}(b^2 + 4N^2 - 1)\sec^2\alpha u + \frac{\alpha^2}{2} bN \sec\alpha u \tan\alpha u + U_m(x) \tag{33a}$$

$$E = \frac{\alpha^2}{8}((1 + 2j)^2). \tag{33b}$$

The Scarf, PT symmetric Scarf and generalized Pöschl–Teller potentials are isospectral potentials. The last six potentials have already been constructed by choosing r as an exponential function. This property implies that these potentials form the same family potentials and they can be obtained from each other by a simple coordinate transformation.

4.4. Eckart family potential

The Eckart family potential can be constructed by introducing $r = \coth\frac{\alpha u}{2}$, $a = 1$. The corresponding potential and eigenvalues are given by

$$V(x) = \frac{\alpha^2}{2} bN \coth \alpha u + \frac{\alpha^2}{2} j(j+1)\,\mathrm{cosech}^2 \alpha u + U_m(x) \tag{34a}$$

$$E = -\frac{\alpha^2}{8}(b^2 + N^2). \tag{34b}$$

The trigonometric form of this potential can be obtained by the choice of

$$r = \cot \frac{\alpha u}{2} \qquad a = 1 \qquad b \to \mathrm{i}b \tag{35}$$

then the potential (14) takes the form

$$V(x) = -\frac{\alpha^2}{2} bN \cot \alpha u + \frac{\alpha^2}{2}(j(j+1))\,\mathrm{cosec}^2 \alpha u + U_m(x) \tag{36a}$$

$$E = -\frac{\alpha^2}{8}(b^2 - 4N^2). \tag{36b}$$

4.5. Hulthen family potential

Another important potential of the quantum mechanics is the Hulthen potential, the choice of $r = \coth \frac{\alpha u}{4}$, $a = 1$ produces the following potential,

$$V = \frac{(j(j+1) + bN/2)\alpha^2\,\mathrm{e}^{-\alpha u}}{2(1 - \mathrm{e}^{-\alpha u})} + \frac{j(j+1)\alpha^2\,\mathrm{e}^{-2\alpha u}}{2(1 - \mathrm{e}^{-\alpha u})^2} + U_m(x) \tag{37a}$$

$$E = -\frac{\alpha^2}{32}(b + 2N)^2. \tag{37b}$$

4.6. Rosen–Morse family potential

The last example in this category is the Rosen–Morse family potential. This potential is isospectral with the Eckart family potential and can be obtained by introducing

$$r = \coth\left(\frac{\alpha x}{2} + \mathrm{i}\frac{\pi}{4}\right) \qquad a = 1. \tag{38}$$

Substituting (38) into (14), we obtain the following potential with the eigenvalues E

$$V(x) = \frac{\alpha^2}{2} bN \tanh \alpha u - \frac{\alpha^2}{2} j(j+1)\,\mathrm{sech}^2 \alpha u + U_m(x) \tag{39a}$$

$$E = -\frac{\alpha^2}{8}(b^2 + 4N^2). \tag{39b}$$

In order to obtain trigonometric form of the Rosen–Morse family potential, we substitute

$$r = -\mathrm{i}\cot\left(\frac{\alpha u}{2} + \frac{\pi}{4}\right) \qquad a = 1 \qquad b \to \mathrm{i}b \tag{40}$$

into (14) and obtain the following potential,

$$V(x) = \frac{\alpha^2}{2} bN \tan \alpha u + \frac{\alpha^2}{2}(j(j+1)) \sec^2 \alpha u + U_m(x) \tag{41a}$$

$$E = -\frac{\alpha^2}{8}(b^2 - 4N^2). \tag{41b}$$

It is obvious that the Eckart, Hulten and Rosen–Morse family potentials can be mapped onto each other by a simple coordinate transformation.

5. Conclusions

In this work we have made a systematic study to obtain the exact solution of the PDM Schrödinger equation within the context of the *su*(1,1) algebra. We have obtained a number of potentials some of which are already known while others are new. Another issue here is the choice of the parameters ρ, η and ε. It has been shown that the exact solvability of the PDM Schrödinger equation is independent of these parameters.

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References

[1] Roy B and Roy P 2002 *J. Phys. A: Math. Gen.* **35** 3961
[2] Milanovic V and Ikanovic Z 1999 *J. Phys. A: Math. Gen.* **32** 7001
Levy-Leblond J M 1995 *Phys. Rev.* A **52** 1845
Levy-Leblond J M 1992 *Eur. J. Phys.* **13** 215
Foulkes W M C and Schluter M 1990 *Phys. Rev.* B **42** 11 505
[3] Serra L I and Lipparini E 1997 *Europhys. Lett.* **40** 667
Barranco M, Pi M, Gatica S M, Hermandez E S and Navarro J 1997 *Phys. Rev.* B **56** 8997
Einevoll G T 1990 *Phys. Rev.* B **42** 3497
Morrow R A 1987 *Phys. Rev.* B **35** 8074
[4] Lévai G 1989 *J. Phys. A: Math. Gen.* **35** 689
[5] Natanzon G A 1971 *Vestn. Leningr. Univ.* **10** 22
Natanzon G A 1979 *Teor. Mat. Fiz.* **38** 146
[6] Cordero P, Hojman S, Furlan P and Gihirardi G C 1971 *Nuovo Cimento* **A3** 807
Wu J, Alhassid Y and Gürsey F 1989 *Ann. Phys. NY* **196** 163
[7] Von Roos O 1983 *Phys. Rev.* B **27** 7547
[8] de Souza Dutra A and Almeida C A S 2000 *Phys. Lett.* A **275** 25
[9] Dekar L, Chetouani L and Hammann T F 1998 *J. Math. Phys.* **39** 2551
[10] Sukumar C V 1986 *J. Phys. A: Math. Gen.* **19** 2229
[11] Bagchi B and Ganguly A 2003 *J. Phys. A: Math. Gen.* **36** L161
[12] Shifman M A 1989 *Int. J. Mod. Phys.* A **4** 2897
[13] Bender C M and Boettcher S 1998 *Phys. Rev. Lett.* **24** 5243
[14] Bagchi B and Roychoudhury R 2000 *J. Phys. A: Math. Gen.* **33** L1